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A Tour Through the Partially Tamed World of Visual Geometry

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Introduction

3D Geometry became an experimental science after the arrival of Cabri 3D in 2004. Here we wish to share our findings under the environment provided by this great software in the form of a tour in the long-forgotten field Visual Geometry. The presentation follows neither the chronological order nor the Euclidean logical order. The tour begins following some kind of visual structure housing the experiments. However, it becomes hopelessly untamed towards the end of the tour. [**Note.** To view dynamic 3D figures, you may click on the images but you need to install Cabri 3D Plug-in, which is available at http://www.cabri.com/download-cabri-3d.html#plugin and use Internet browser to view dynamic 3D figures.]

Wireframe and Cage

The underlying design structure of a 3D geometric object is better understood when some parts are displayed as wireframe models. Thinking the hollow faced polygon as a cage, the dual pairs of regular polyhedrons (Fig. 1-2) have an apt description that each polyhedron is caged inside its dual.

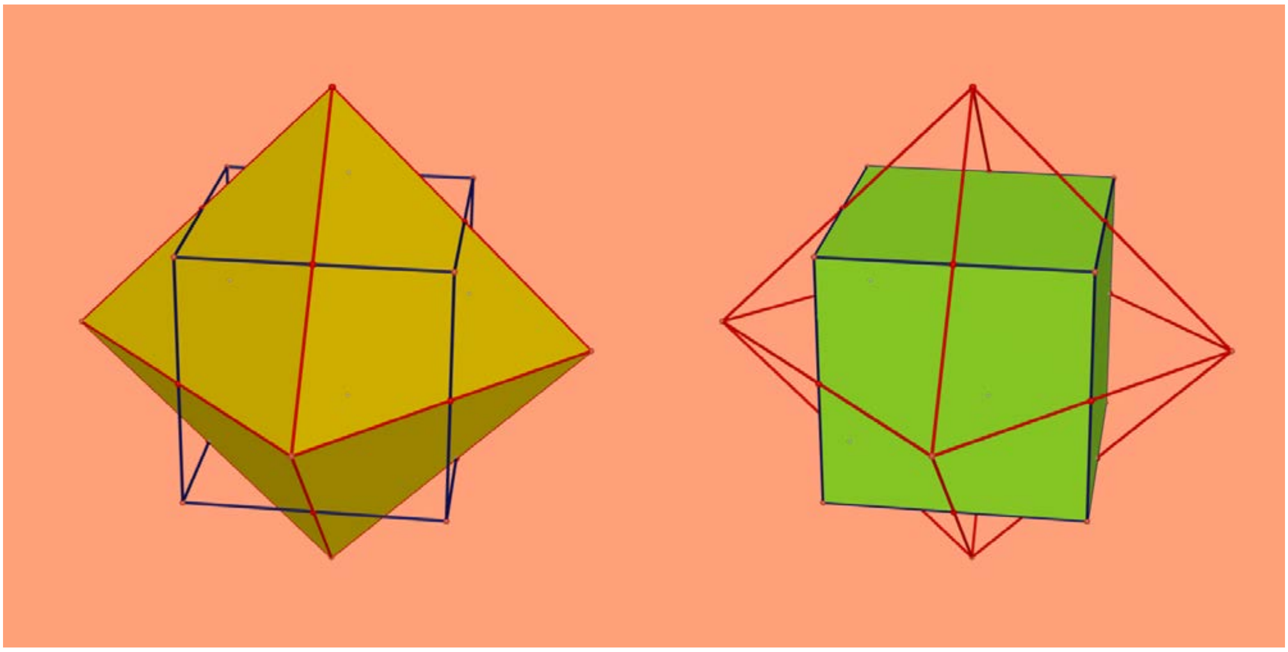


Fig. 1 Octahedron Caged inside the Cube　　*Fig. 2 Cube Caged inside Octahedron*

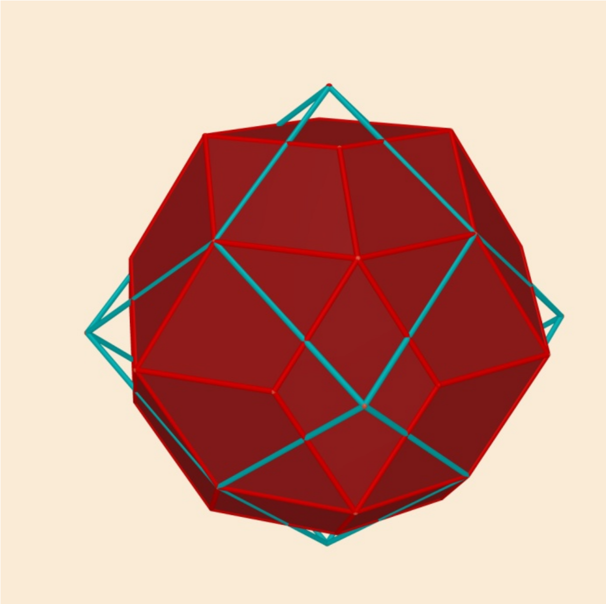


Fig. 3 Rhombic Triacontahedron Caged Inside a Rhombic Dodecahedron

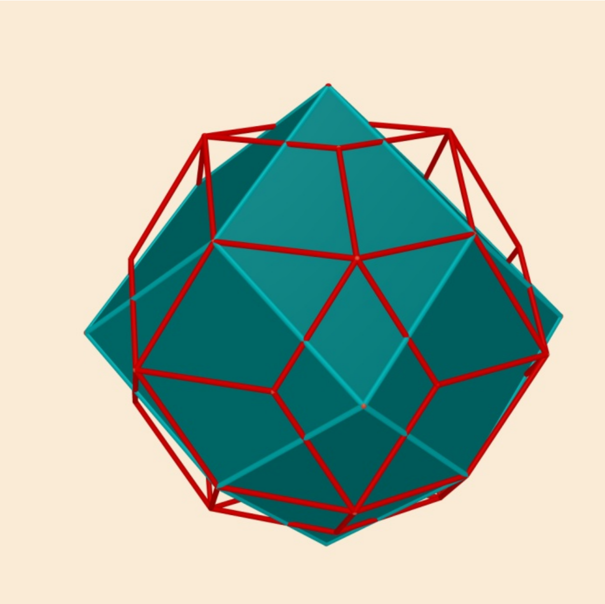


Fig. 4 Rhombic Dodecahedron Caged Inside a Rhombic Triacontahedron

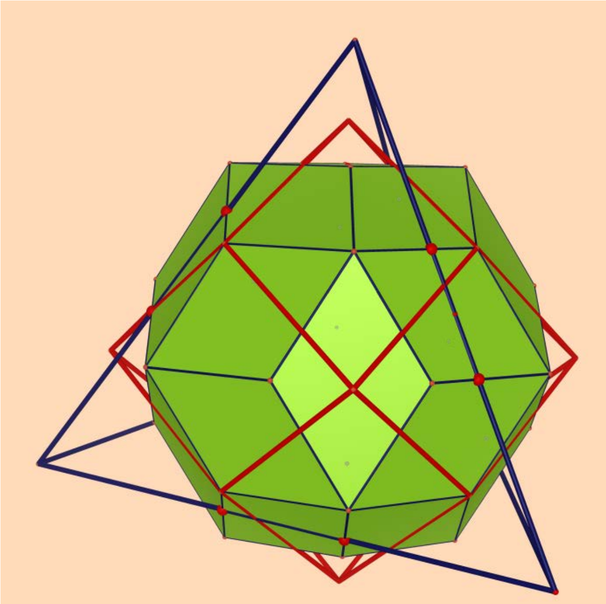


Fig. 5 Rhombic Triacontahedron Caged Inside Tetrahedron

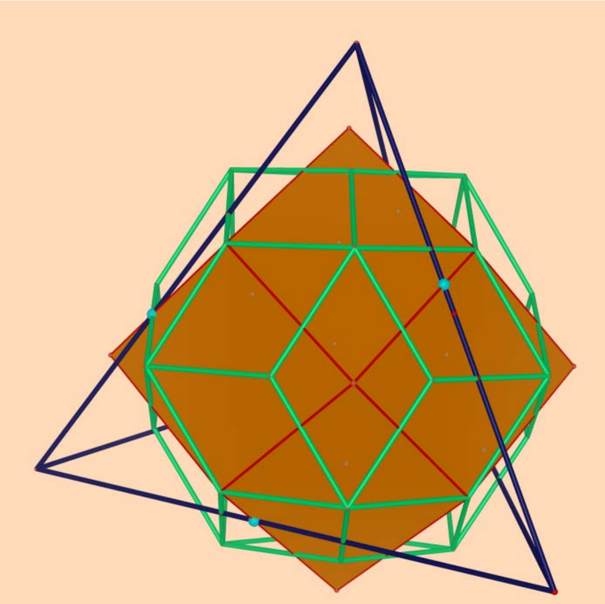


Fig. 6 Rhombic Dodecahedron Caged Inside Tetrahedron

Fig. 3-6 show that there are plenty to be learned from caging one polyhedron inside another.

Yin-Yang Principle

Hollow faces polyhedrons may have inspired the Yin-Yang principle in fashion designs. The five Platonic solids form the corner stones for Western philosophy. Oriental philosophy, on the other hand, is largely based on Yin-Yang Principle. One can start a research on the relationships between Platonic solids and the Yin-Yang Principle with an entertaining drawing of these five uncommon models: Cubohemioctahedron, Octahemioctahedron, Small Dodecahemidodecahedron, Small Icosihemidodecahedron and Tetrahemihexahedron (Fig. 7-11).

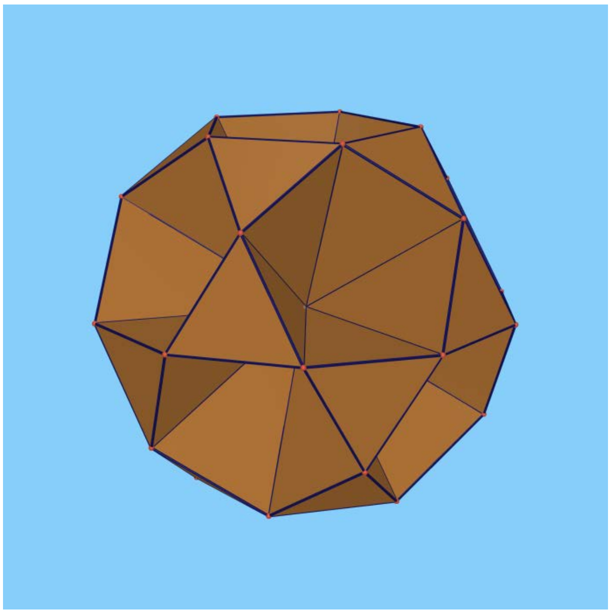


Fig. 7 Small Dodecahemidodecahedron

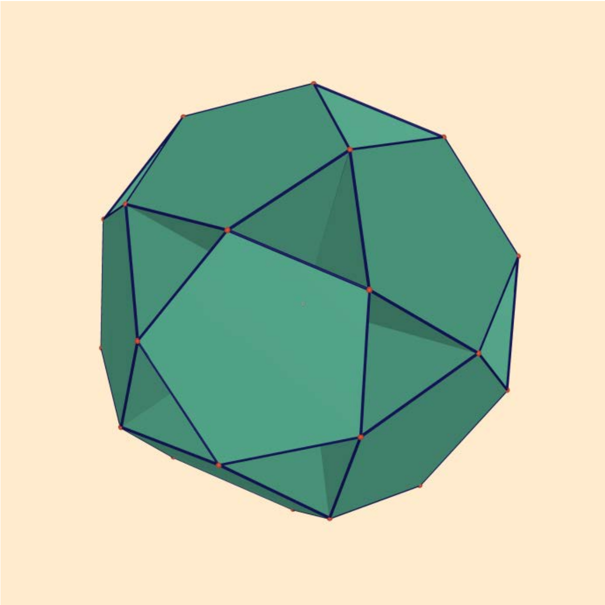


Fig. 8 Small Icosihemidodecahedron

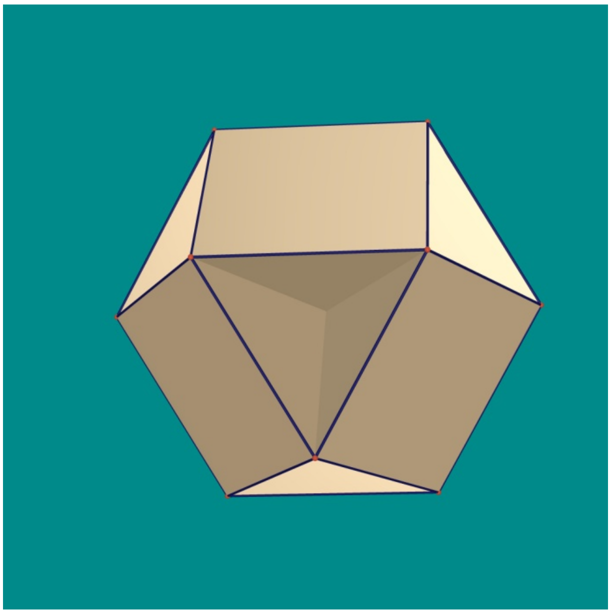


Fig. 9 Cubohemioctahedron

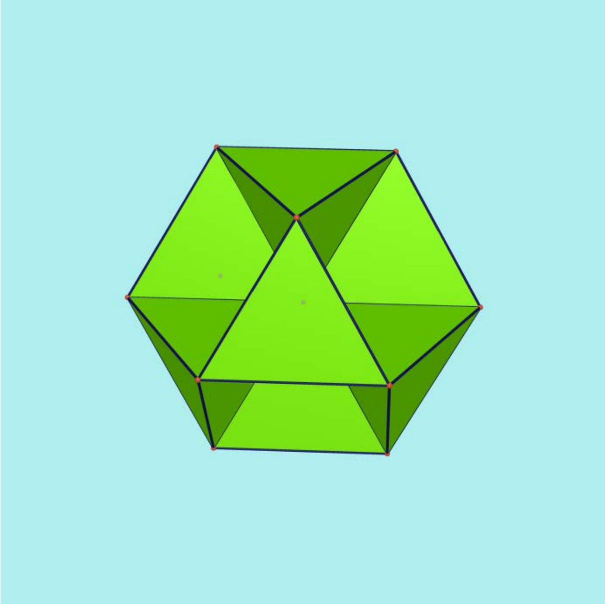


Fig. 10 Octahemioctahedron

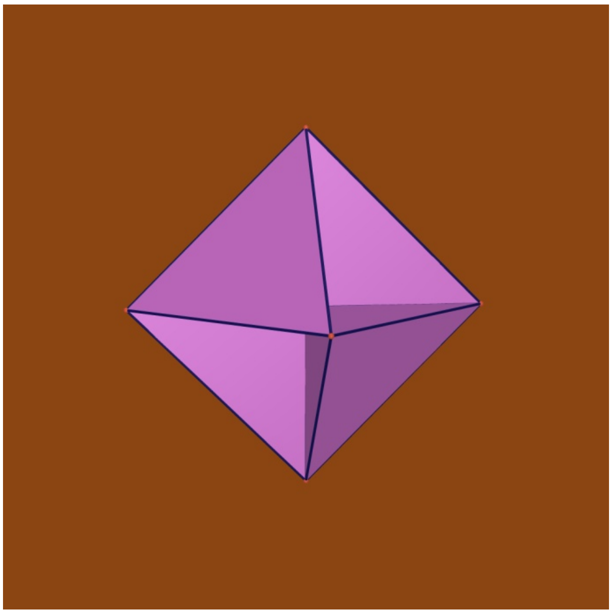
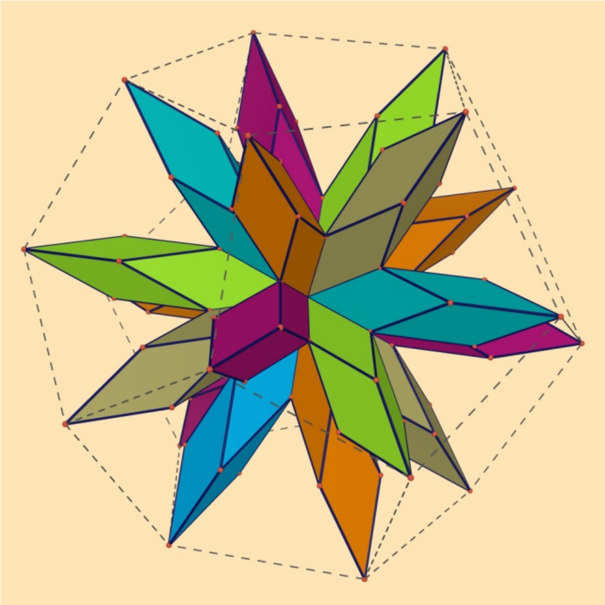


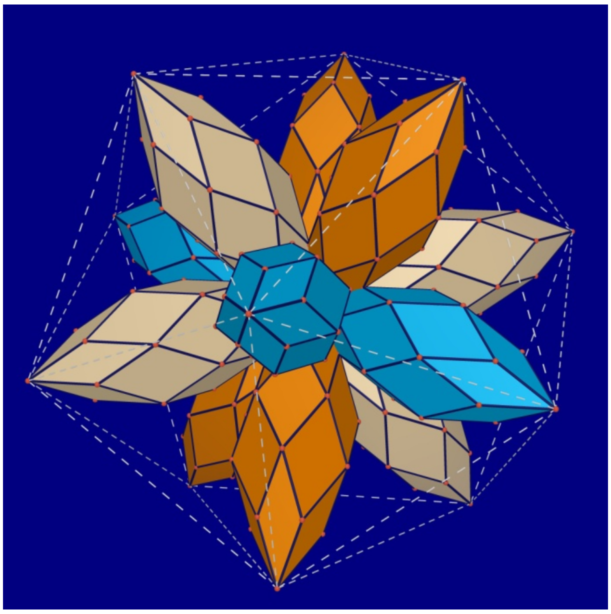
Fig. 11 Tetrahemihexahedron

Crystals Grown on Yin-Yang Models

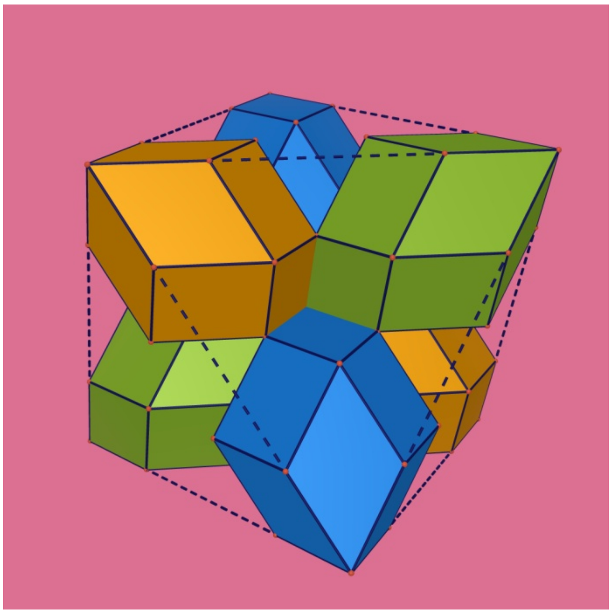
Each block used in Yin-Yang model is able to grow into a basic rhombic-faced block. Assembling these rhombic blocks together the same way as the Yin-Yang models, we end up with a collection of aesthetic models that resemble crystalline solids.



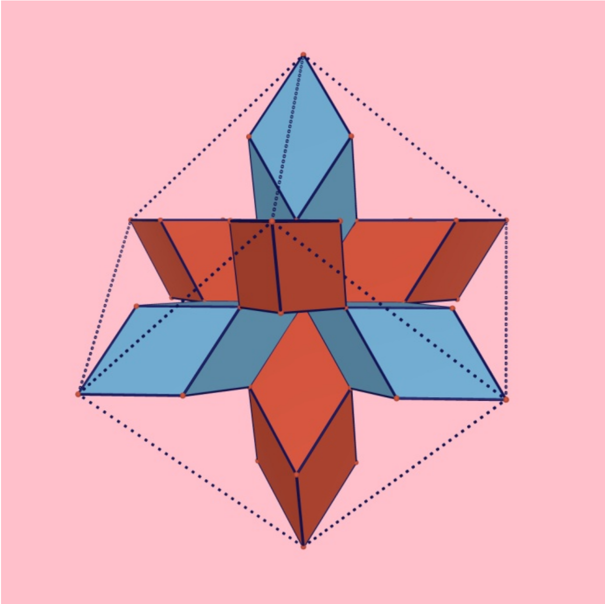
120-Faced Rhombic Polyhedron Grown from Small Dodecahemidodecahedron



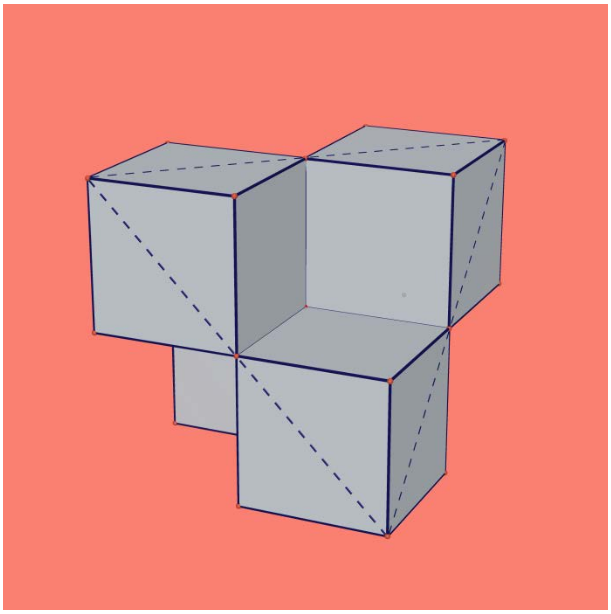
240 Faced Polyhedron Grown from Small Icosihemidodecahedron



72-Faced Polyhedron Grown from Cubohemioctahedron



48-Faced Polyhedron Grown from Octahemioctahedron



24-Faced Rhombic Polyhedron Grown from Tetrahemihexahedron

Direct Generation of Faces

As in generating the dodecahemidodecahedron, all regular triangular faces in a icosidodecahedron can be directly generated by taking solely the successive line reflections of one single face of a regular icosahedron across the three most symmetric adjacent edges (Fig. 12). To highlight the rich symmetry and the combinatorial properties hidden in this configuration, the 20 triangles are painted with five colors, with the centers of the four triangles of the same color occupying the vertices of some hidden regular tetrahedron.

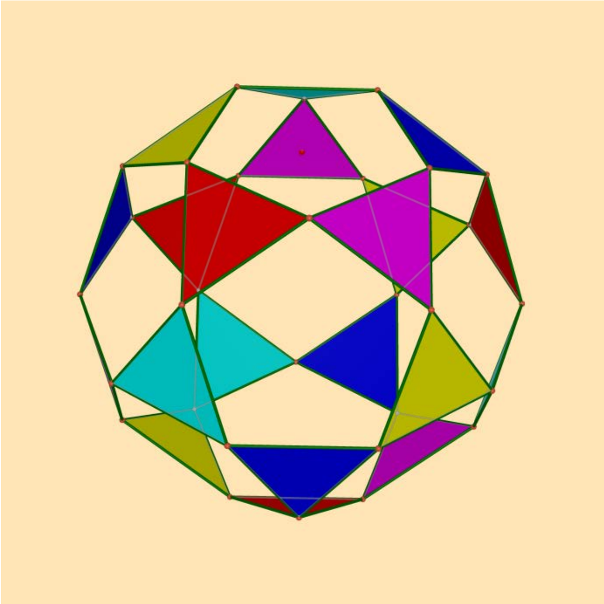


Fig. 12 Triangular Faces of Icosidodecahedron

The dual object of the last pattern consists of 12 regular pentagons (Fig.13). The configuration could also be generated by taking line reflections only. The most mathematically pleasing way to paint the pentagons is to use three colors so the axis of the four pentagons of the same color belong to exactly one of three mutually perpendicular planes .

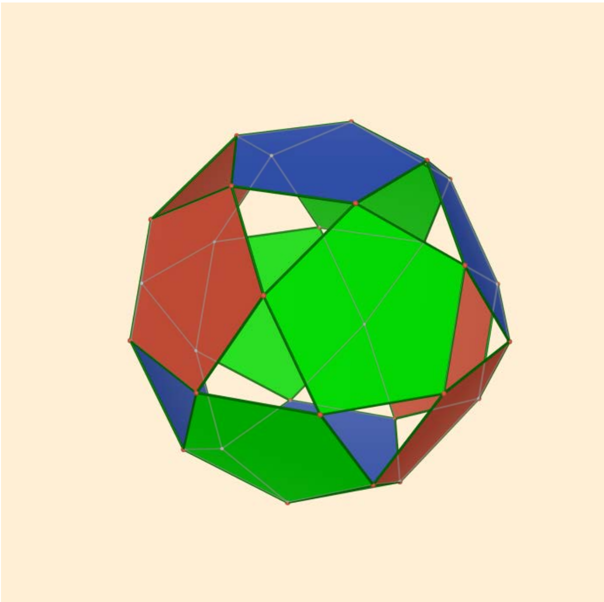


Fig. 13 Pentagonal Faces of Icosidodecahedron

HINT: Always Leave Something Empty

By now you are hypnotized with the message "Always Leave Something Empty". This magic recipe will always arouse curiosity in a surprising way. The following illustration (Fig. 14), with the intermediate steps undisplayed, challenges the reader to draw a hyperbolic paraboloid as a ruled surface situated above a circular disc.

Anything left empty? Einstein said the space and the time are one and the same thing.

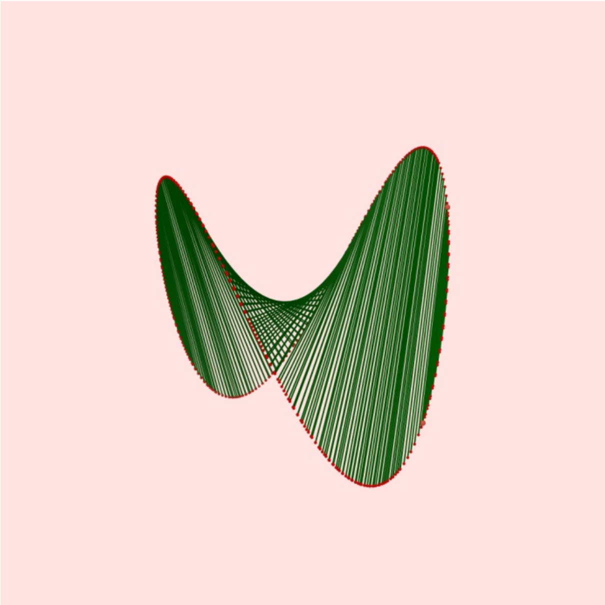


Fig. 14 Hyperbolic Paraboloid Drawn as a Ruled Surface This surface enjoys the rare property of being a doubly-ruled surface (Fig. 15):

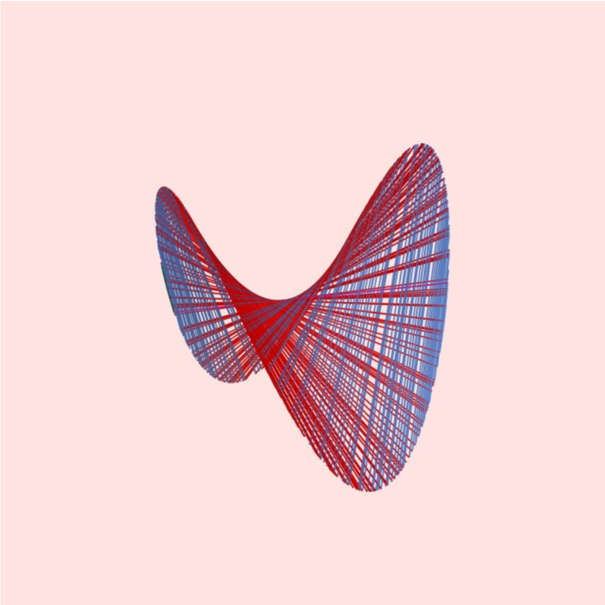


Fig. 15 Hyperbolic Paraboloid Drawn as a Doubly Ruled Surface

Villarceau Circle, Half-Torus and Hula Hoop

Given an arbitrary point on a torus, it may be visually unconvincing that FOUR circles can be drawn through it! The two unusual ones are the Villarceau circles. An unusual half-torus (Fig. 16) can therefore be generated by sweeping out the Villarceau circle making a half-turn around the axis of the torus. Unlike the hyperbolic paraboloid, this surface shies away from most Calculus texts and enjoys having a pair of linked circles in space as components of its boundary.

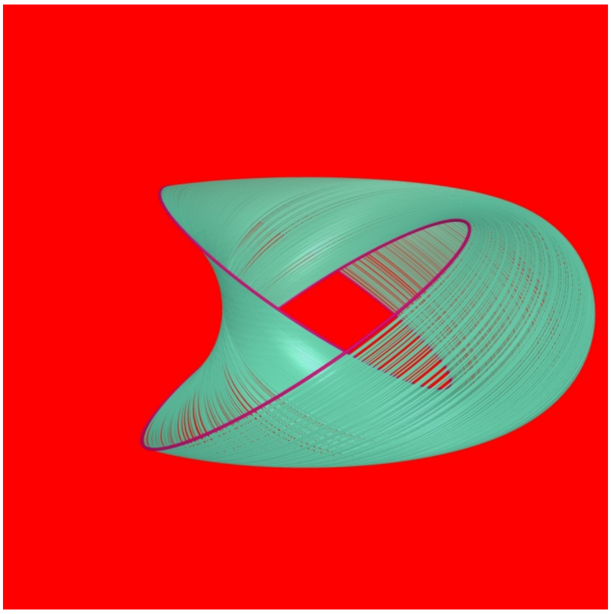


Fig.16-A Half-Torus Generated by Villarceau Circle

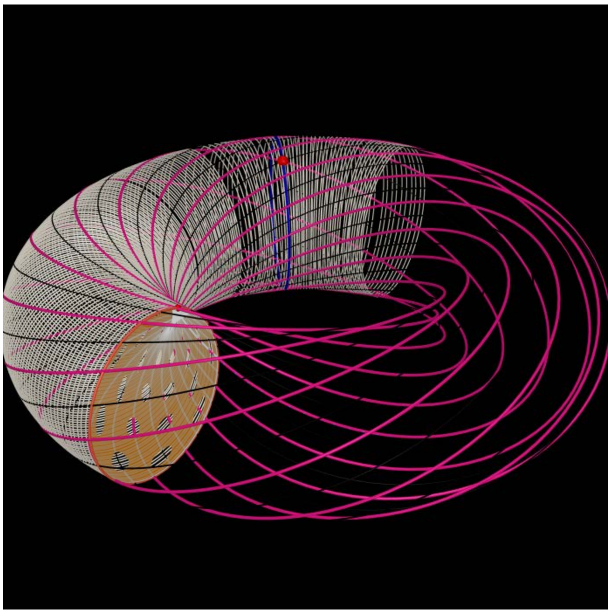


Fig. 16-B 24 Villarceau Circles

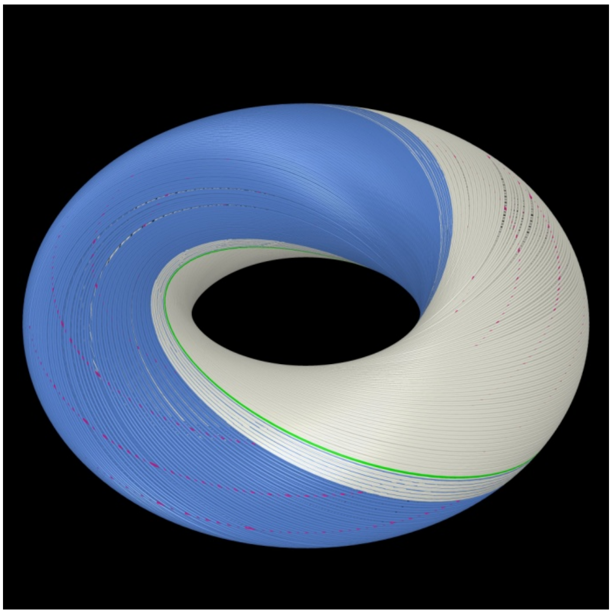


Fig. 16-C Half-Tori with Villarceau Circle as Boundary

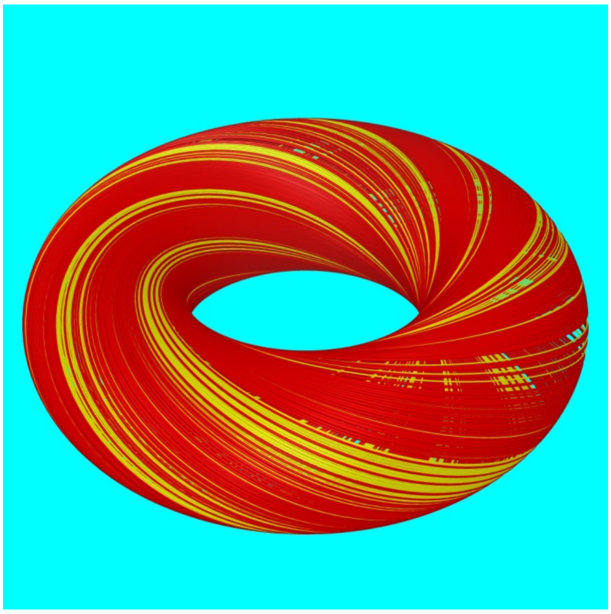


Fig. 16-D Design Based on Villarceau Circles

The animation below (Fig. 16-E) suggests the ideal body shape for a Hula dancer: the waist should coincide with parts of a torus (blue arcs) and the hula hoop (red circle) should be custom-built to have the same radius as the associated Villarceau circle. The maximum contact between the waist and the hoop is reached If the hoop should rotate about the axis of the torus the same way as the Villarceau circle.

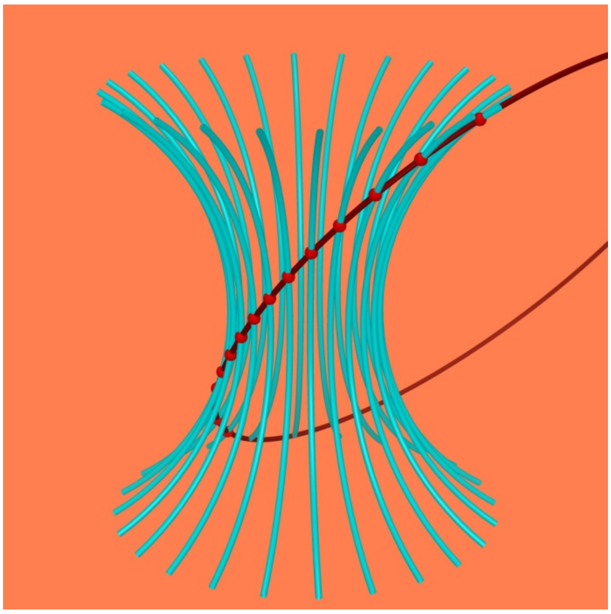


Fig. 16-E Ideal Body Shape for Hula

Commercial Break: This tour promotes environmental-friendliness whenever possible.

Ribbed Sculpture

In 1858 August Ferdinand Möbius made his name in Topology by designing his famous one-sided strip. Inspired by the construction of this non-orientable surface, Charles Owen Perry made his name in Visual Art in 1985 with the creation of a large-scale wire-frame sculpture "Solstice". It was formed by rotating a triangle by two-thirds twists as it rotates around the ring (Fig. 17). There are many profitable ribbed sculptures like this one waiting to be designed.

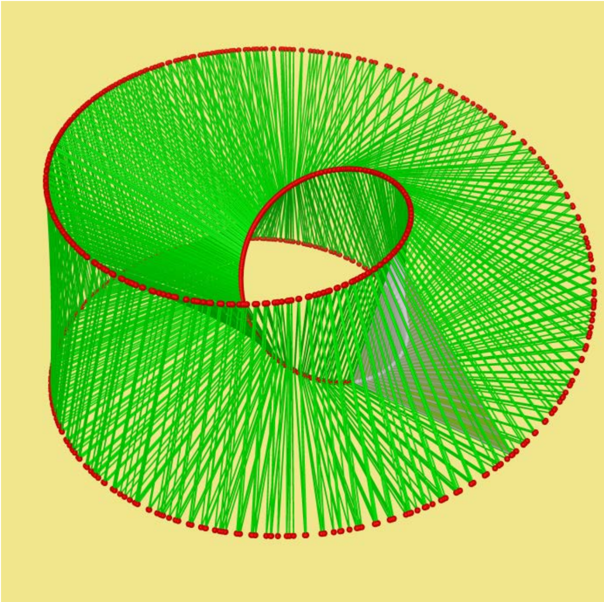


Fig. 17 Two-Thirds Twist Triangular Torus

Rotating Rings of Tetrahedra

Kaleidocycles (also known as Rotating Rings of Tetrahedra [2] and Flexahedron [4]) seemed to have been documented in Max Brückner's Vielecke und Vielflache in 1900 [1]. (Concrete geometric models were truly alive then.) Toys like these show how odd-shaped couples formed by tetrahedrons confined in a torus can still dance so gracefully (Fig. 20). The crucial part of the construction (Fig. 18-19) is to to calibrate the rotating tetrahedron so the two nonadjacent edges always slide on two preset planes making a planar angle pi / n. Once this setting is established, the other tetrahedrons comprising the 3D linkage are obtained by taking successive ping-pong styled planar reflections through the two planes.

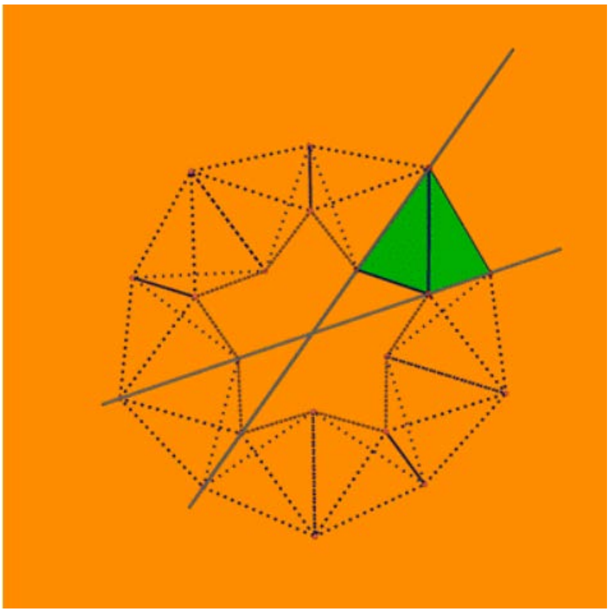


Fig. 18 Initial Stage Setting of Animation for Kaleidocycle with 10 Regular Tetrahedrons

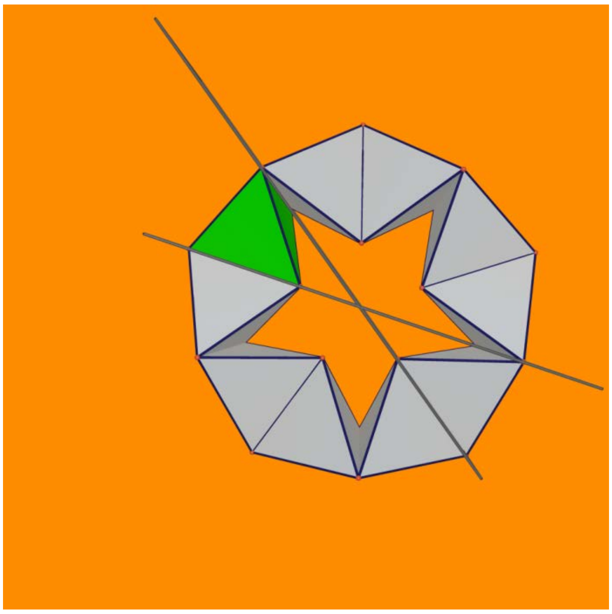


Fig. 19 Ping-Pong Relections

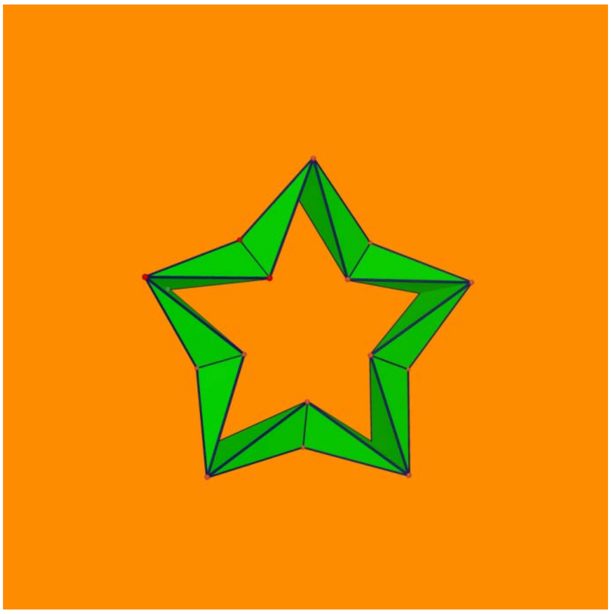


Fig. 20 3D Linkage formed by Rotating Ring of Non-Regular Tetrahedra

Concentric Polyhedrons

Mathematics was reformulated and organized by puritans of 20th Century according to the "structures". Comparison of structures is technically expressed in terms of morphisms, mappings dignified by the abstract nonsense specialists. This modern machinery, however, enables us to solve a certain type of dissection problems which were considered "hard" in classical geometry, by means of linear perspective, the mappings compatible with structures given by the wireframes.

Hard Problem 1: Dissect a cube into six mutually congruent pieces non-trivially.

Solution: Instead of finding only one dissection we aim to find a whole one-parameter family of such dissections. Consider the central perspectivity of (faces of) a cube upon itself as its wireframe rotates about a fixed main diagonal. The perspective image of the each face of the rotating cube (Fig. 21) coincides with the image given by its neighboring face after a rotation of 120 degrees in either directions. A similar property holds when 120-degree rotation is replaced by taking central point reflection. The perspective images of all six rotating faces altogether cover the stationary cube. Hence the convex hull determined by the center and the image of a face form one of six mutually congruent pieces in the required dissection.

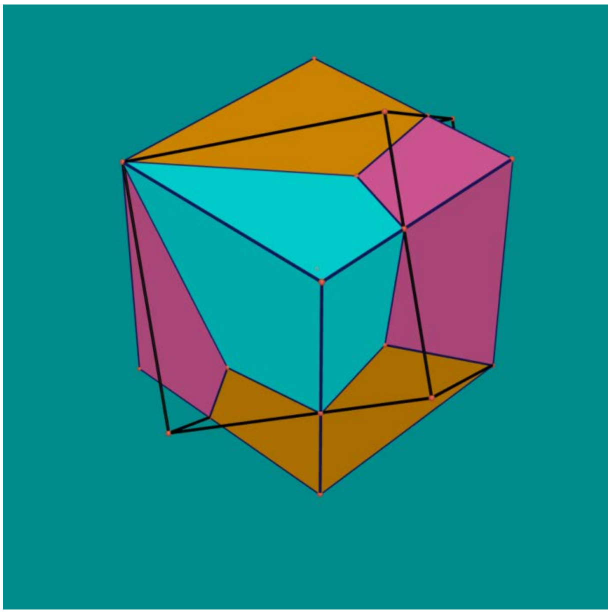


Fig. 21 Central Projection of a Rotating Cube onto a Fixed Cube

Hard Problem 2: Dissect a regular tetrahedron into four mutually congruent pieces non-trivially.

Solution: Overcoming the difficulty of superimposing the left hand with the right hand mathematically, Problem 2 is no harder than Problem 1. The construction of Problem 2 follows steps similar to Problem 1, except the transformation "pointwise central symmetry" is replaced by a rotoreflection, the composition of a rotation of the pointwise central symmetric image followed by a 90 degrees rotation with respect to the axis of symmetry.

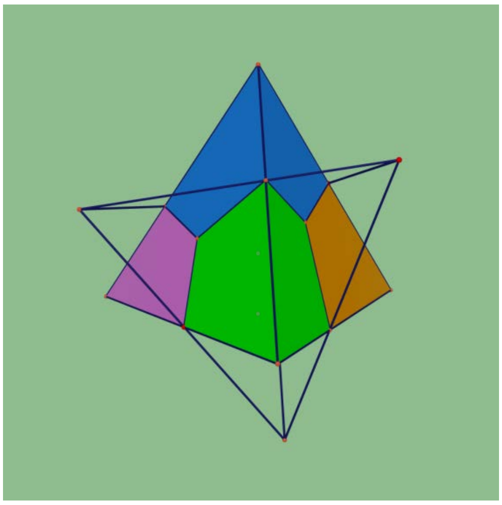


Fig. 22 Central Projection of a Tetrahedron onto a Fixed Tetrahedron

Construction of the IOTCD Chain in Eleven Steps

Here comes the exotic part of the tour: we are to construct all five Platonic solids resulting in a chain consisting of Icosahedron, Octahedron, Tetrahedron, Cube, Dodecahedron in ascending order (Fig. 23). Here the full polyhedrons are partially ordered by set-inclusion and the name chain requires any two be comparable. The construction takes only 11 steps, thanks to Cabri 3D with which a regular polyhedron is determined when one of its (oriented) faces is specified.

- (Step 1) Construct the regular icosahedron RI. This is the solid polyhedron in Fig. 23.
- (Step 2 - Step 3) Construct the smallest regular tetrahedron RT containing RI, (Fig. 24). Hint: One face of RT has the same center as the face defining RI and has one vertices lying on the extension of one edge of RI.
- (Step 4 - Step 5) Construct the unique octahedron RO squeezed between RI and RT. Hint: Midpoints of all edges of RI comprise all vertices of RO. (Step 6 - Step 8) Construct the regular dodecahedron RD having all four vertices of RT as its vertices. Hint: Each face of RD has the same axis as RI.
- (Step 9 - Step 11) Construct the unique cube squeezed between RD and RT. (E face of the cube has the same axis as RO.)

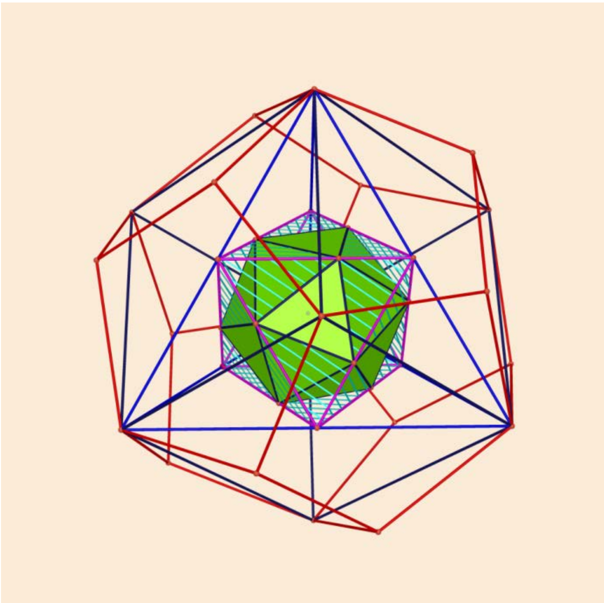


Fig. 23 Construction of the IOTCD Chain in Eleven Steps

Intersection and Convex Hull of Tetrahedrons

At the end of Step 3 in the IOTCD construction, we have created the tightly configured RI-RT pair (Fig. 2) in which each face of RT contains a face of RI.
By rotating RT successively with respect to a fixed diagonal of RI by 72,144, 216, 288 degrees we obtain altogether five tetrahedrons with the property that each face of RI is contained in exactly one face of the five RT. This last configuration illustrates (Fig. 25) the mildly challenging fact that the if five tetrahedrons have a regular dodecahedron as their convex hull, then their common intersection is a regular icosahedron.

Intersection and Convex Hull of Octahedrons

At the end of Step 5 of of the IOTCD construction we have created the tightly configured RI-RO pair (Fig. 4) in which each face of RO contains one face of RI.

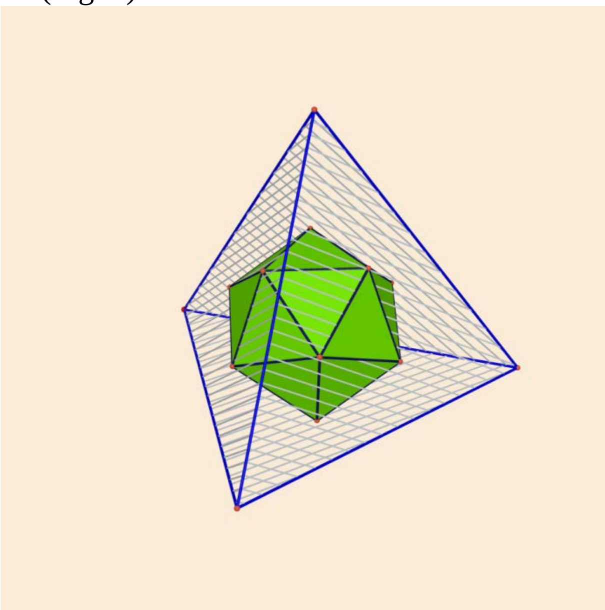


Fig. 24 Tightly Configured RI-RT Pair

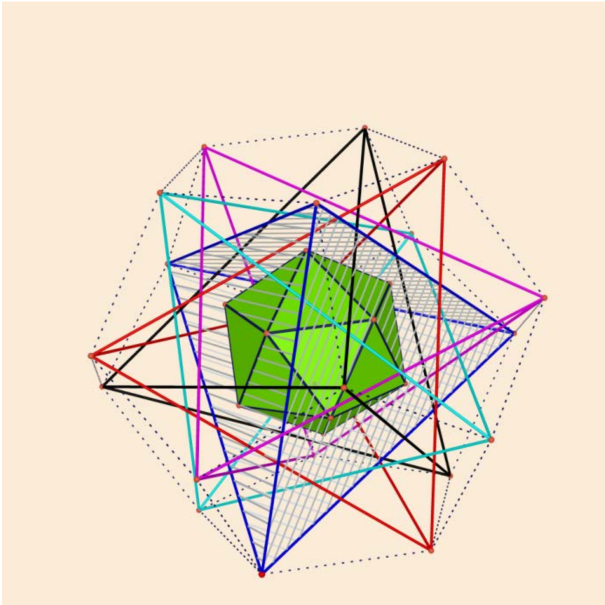


Fig. 25 Intersection and Convex Hull of Five Regular Tetrahedrons

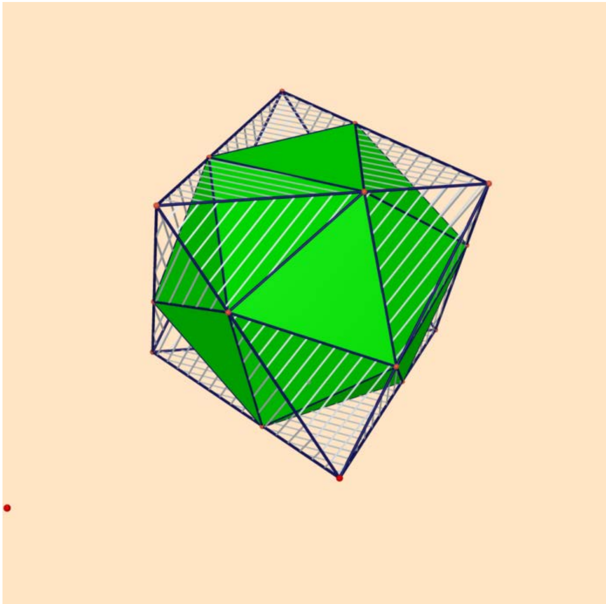


Fig. 26 Each Face of RO Supports RI

By rotating RO successively with respect to a fixed diagonal of RI by 72,144, 216, 288 degrees we obtain altogether five octahedrons with the property that each face of RI appears in exactly two faces of the five RO. This configuration illustrates (Fig. 27) the fact that the if five octahedrons have a regular icosidodecahedron as their convex hull, then their intersection is a regular icosahedron.

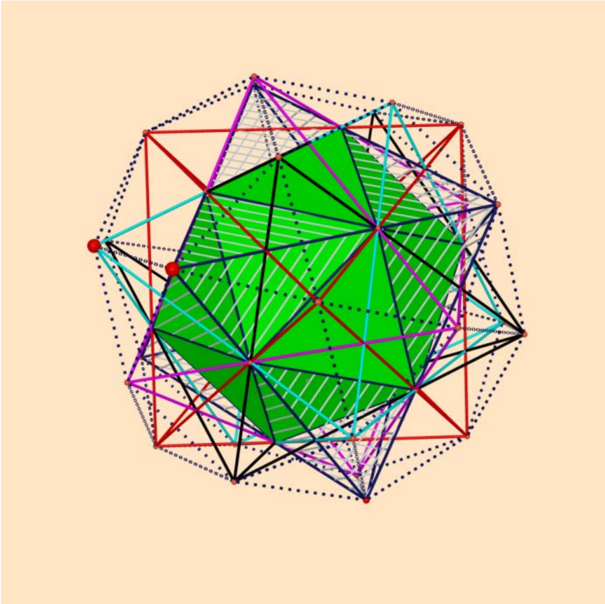


Fig. 27 Intersection and Convex Hull of Five Regular Octahedrons

We conclude from Fig. 25 and Fig. 27 that a regular icosahedron can be expressed both as an intersection of five regular tetrahedrons as well as an intersection of five regular octahedrons!

Representation of Polyhedrons as Intersection of Basic Ones

There is no known method to carry out the factorization for a large integer quickly. Its complexity is the basis of the assumed security of some public key cryptography algorithms, such as RSA. The problem of expressing a polyhedron as an intersection of basic ones may be facing the same fate. Here are three examples (Fig. 28-30), all carried out by brute force.

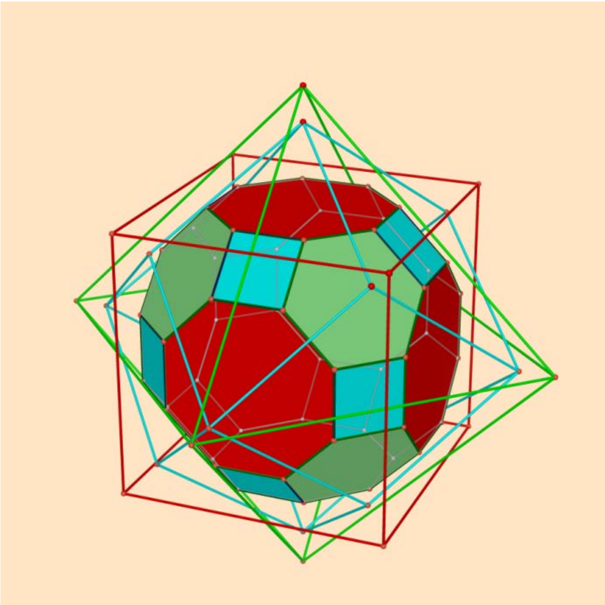


Fig. 28 Great Rhombicuboctahedron Shown as An Intersection of Rhombic Dodecahedron, Ocahedron and Cube

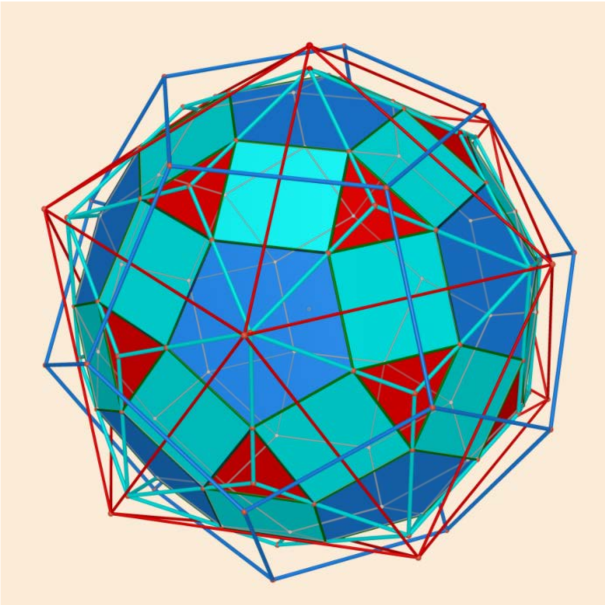


Fig. 29 Great Rhombicosidodecahedron Shown as An Intersection of Rhombic Triacontahedron, Dodecahedron and Icosahedro

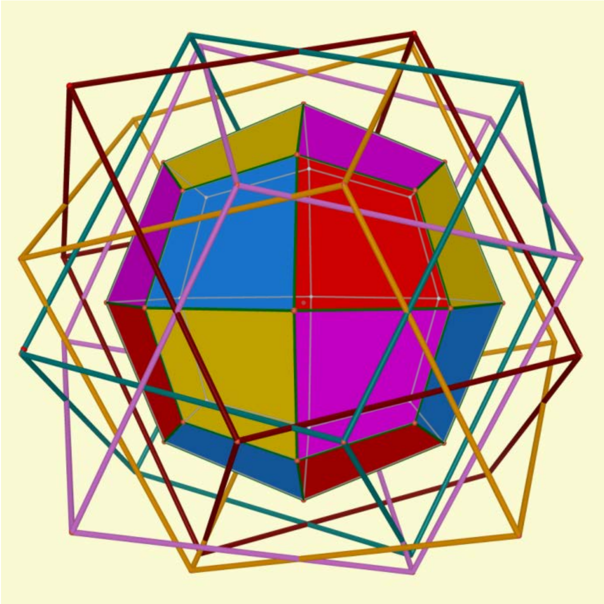


Fig. 30 Deltoidal Icositetrahedron Shown as Intersection of Rhombic Hexahedrons

Untamed Areas

By shrinking/enlarging the polyhedrons by appropriate factors, there are at least 5! = 120 different configurations associated with IOTCD Chain. Here are three such possibilities (Fig. 31-33):

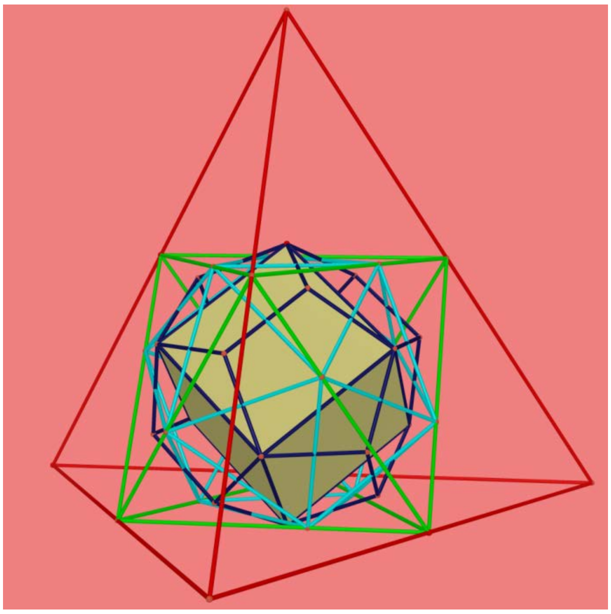


Fig. 31 Variation 1 of IOTCD

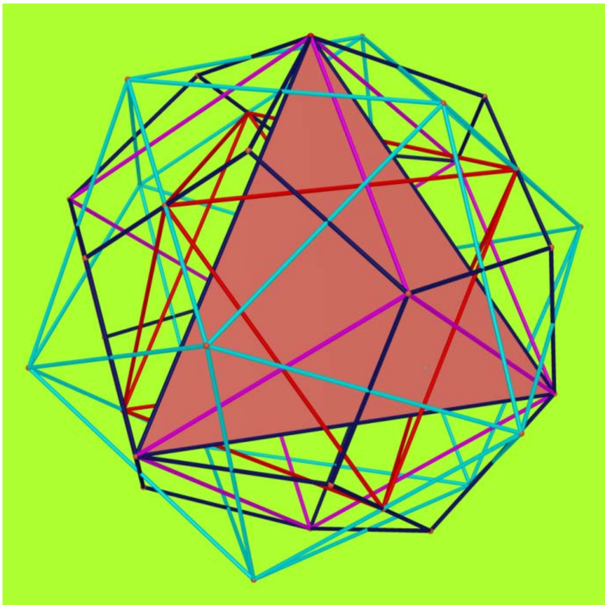


Fig. 32 Variation 2 of IOTCD

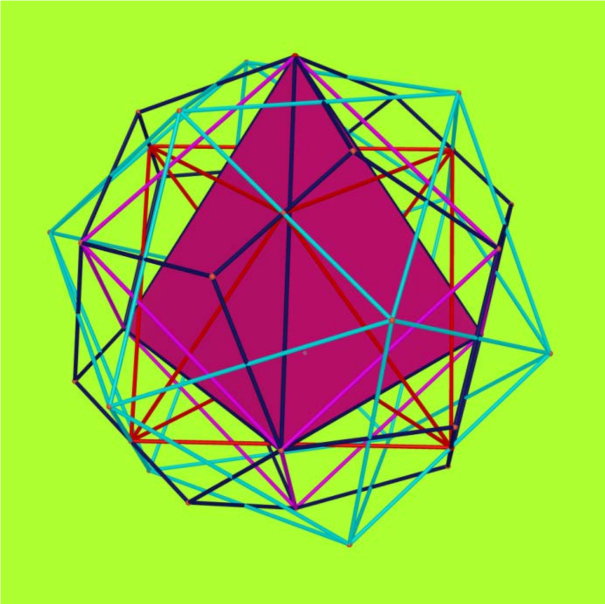


Fig. 33 Variation 3 of IOTCD

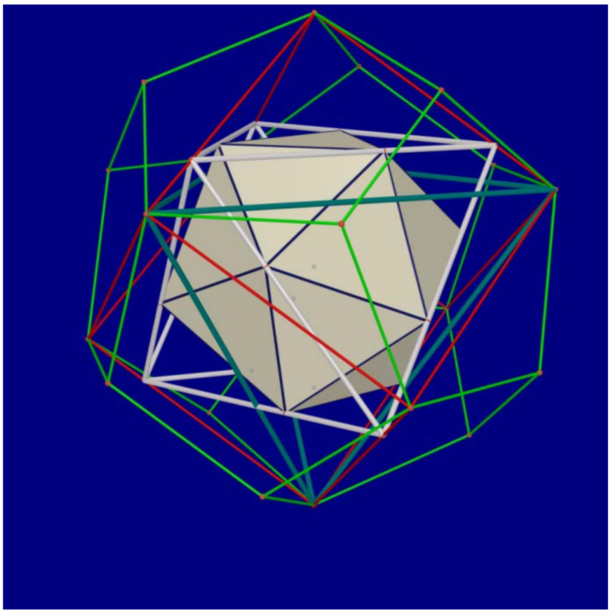


Fig. 34 Something Outside of IOTCD Camp

Note that the example given in Fig. 34 does not belong to the family of 120 variations of IOTCD.

For those planning to display something innovative for their museum or souvenir shop, the exotic constructions should provide plenty of inspiration. Everyone taking part in this tour would love to be your customer.

Note. Author will constantly add more information after this publication at this URL: http://140.114.32.248/d2/A%20Tour/A%20Tour.html.

Conclusion

An area of mathematics is alive if refreshing ideas are formed. In this tour we wanted to show how Visual Geometry becomes alive again. We also wish to express our thanks to the creators of Cabri 3D. You have set the quality standard for mathematical software by not including a large number commands, but by crafting out the highly efficient environment for mathematical dreaming.

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[1] Max Brückner, Vielecke und Vielflache, 1900 and 1906. Internet Archive Download Link Appendix Link

[2] Brian Bolt, More Mathematical Activities, CUP, 1985, 20-22. Google Book Link.

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